

**RECONSTRUCTING THE DENSITY OPERATOR  
BY USING GENERALIZED FIELD QUADRATURES**

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**Abstract**

The Wigner function for one and two-mode quantum systems is explicitly expressed in terms of the marginal distribution for the generic linearly transformed quadratures. Then, also the density operator of those systems is written in terms of the marginal distribution of these quadratures. Some examples to apply this formalism, and a reduction to the usual optical homodyne tomography are considered.

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# 1 Introduction

The homodyne measurement of an electromagnetic field gives all the possible combinations of the field quadratures belonging to the group  $O(2)$ , by just varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of the Wigner function, or any other quasi-probability used in quantum optics. In the work [1] it was shown, indeed, that the rotated quadrature distribution may be expressed in terms of the Wigner function [2] as well as in terms of the Husimi  $Q$ -function [3] and the Glauber-Sudarshan  $P$ -representation [4], [5]. The result of Ref. [1] was based on the observation that relations of the density matrix in different representations to the characteristic function, intended as the mean value of the displacement operator creating a coherent state from the vacuum [4], [6], may be rewritten as relations where the characteristic function is the mean value of the rotated quadrature phase. It gave the possibility to express the Wigner function in terms of the marginal distribution of homodyne outcomes through the tomographic formula. The essential point of the obtained formula is that the homodyne output distribution may be measured and the corresponding inverse Radon transform produces the Wigner distribution function, which is referred as the optical homodyne tomography [7]. The density matrix elements, in some representations, can also be obtained by avoiding the Wigner function and then the Radon transform [8], [9], [10].

The tomographic formula of Ref. [1] has been presented in an invariant form in Ref. [10]. In that work the system density operator was expressed as convolution of the marginal distribution of the homodyne output and a kernel operator. In Ref. [11] the tomographic formula of Ref. [1] was extended to express the Wigner function in terms of the marginal distribution of the generalized quadrature obtained through canonical transforms belonging to the Lie group  $ISp(2, R)$ . Essentially this generalized quadrature differs from the one used in the standard tomography since it depends upon three real parameters instead of only one (really, a scaling transformation is also present in the homodyne measurement because position and momentum have different units, but

the scaling cannot be independently controlled of the rotation).

The aim of this paper is to obtain the invariant form of the density operator of the quantum system in terms of the discussed [11] marginal distributions for the generalized quadratures, which is analogous to Ref. [10], and to extend the analysis to the multimode case.

Recently the quantum tomography was also considered to investigate material systems [12] besides the optical ones, thus our alternative approach could be useful for a better characterization of their quantum state since it allows to get overcomplete information.

The structure of the article is the following. In Section 2 we derive the density operator as a convolution of the marginal distribution for the generic linear quadrature and a kernel operator, and discuss the properties of the obtained formula in number state basis, coherent states basis and coordinate representation. In Section 3 we consider the obtained formula applied to some examples of interest, such as the ground state of a harmonic oscillator, the Schrödinger-cat state, and the thermal equilibrium state of an oscillator. In Section 4 we contrast the obtained formula with the tomographic one for the density operator of Ref. [10]. In Sections 5 and 6 the approach will be extended to the multimode case. Some applications of the developed formalism as well as practical implementation and further developments will be discussed in Sections 7 and 8.

## 2 Density Matrix and Marginal Distribution

Let us consider the quadrature observable  $\hat{X}$ , which is a generic linear form in position  $\hat{q}$  and momentum  $\hat{p}$

$$\hat{X} = \mu\hat{q} + \nu\hat{p} + \delta \quad (1)$$

with  $\mu, \nu, \delta$  real parameters (their physical meaning will be discussed later on), then it is possible to get the density matrix elements from the marginal distribution avoiding the evaluation of the Wigner function as an intermediate step. For this purpose we start from a well known [6] representation of the density operator

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} W(\alpha) \hat{T}(\alpha) \quad (2)$$

where the Wigner function  $W(\alpha)$  is a weight function for the expansion of the density operator in terms of the operator  $\hat{T}(\alpha)$  which is defined as the complex Fourier transform of the displacement operator  $\hat{D}$

$$\hat{T}(\alpha) = \int \frac{d^2\xi}{\pi} \hat{D}(\xi) \exp(\alpha\xi^* - \alpha^*\xi). \quad (3)$$

Following the lines of Ref. [11] we may write the marginal distribution  $w$  for the generic quadrature of Eq. (1) as

$$w(X, \mu, \nu, \delta) = \int e^{-ik(X-\mu q-\nu p-\delta)} W(q, p) \frac{dk dq dp}{(2\pi)^2}, \quad (4)$$

where the canonical coordinate  $q$  and  $p$  are related to  $\alpha$  by

$$\alpha = \frac{q + ip}{\sqrt{2}}. \quad (5)$$

Eq. (4) shows that  $w$  is function of the difference  $X - \delta = x$ , so that it can be rewritten as

$$w(x, \mu, \nu) = \int e^{-ik(x-\mu q-\nu p)} W(q, p) \frac{dk dq dp}{(2\pi)^2}; \quad (6)$$

and by means of the Fourier transform of the function  $w$  one can obtain the relation

$$W(q, p) = (2\pi)^2 z^2 w(z, -zq, -zp) \quad (7)$$

where  $-zq, -zp, z$  are the conjugate variable to  $\mu, \nu, x$  respectively. It is worth to remark that in this case the connection between the Wigner function and the marginal distribution is simply guaranteed by means of Fourier transform instead of the Radon one. Then inserting Eq. (7) into Eq. (2), expressing the marginal distribution in terms of the Fourier transform, we have:

$$\hat{\rho} = \int \frac{dq dp}{(2\pi)^2} \hat{T}(q, p) \int dx d\mu d\nu \quad z^2 w(x, \mu, \nu) e^{-izx + i\mu zq + i\nu zp}, \quad (8)$$

or, in a compact form

$$\hat{\rho} = \int dx d\mu d\nu \quad w(x, \mu, \nu) \hat{K}_{\mu, \nu}, \quad (9)$$

where the kernel operator  $\hat{K}_{\mu, \nu}$  is given by:

$$\begin{aligned} \hat{K}_{\mu, \nu} &= \int \frac{dq dp}{(2\pi)^2} z^2 e^{+i\mu zq + i\nu zp - izx} \hat{T}(q, p) \\ &= \frac{1}{2\pi} z^2 e^{-izx} e^{-\frac{z}{\sqrt{2}}(\nu - i\mu)\hat{a}^\dagger} e^{\frac{z}{\sqrt{2}}(\nu + i\mu)\hat{a}} e^{-\frac{z^2}{4}(\mu^2 + \nu^2)} \end{aligned} \quad (10)$$

where we have used Eq. (3) and the Baker-Hausdorff formula. The fact that  $\hat{K}_{\mu,\nu}$  depends on the  $z$  variable as well (i.e. each Fourier component gives a selfconsistent kernel) shows the overcompleteness of information achievable by measuring the observable of Eq. (1). Thus, Eq. (9) can be useful to completely determine the properties of the system, i. e. the density operator, from the probability distribution of the experimental data; outcomes of a set of measurements such as described in Ref. [11].

Let us now consider the expression of the kernel operator (10) in various representations. First, in the coherent states basis we have

$$\langle \alpha | \hat{K}_{\mu,\nu} | \beta \rangle = \frac{1}{2\pi} z^2 e^{-izx} e^{-\frac{z}{\sqrt{2}}(\nu - i\mu)\alpha^*} e^{\frac{z}{\sqrt{2}}(\nu + i\mu)\beta} e^{-\frac{z^2}{4}(\mu^2 + \nu^2)} e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta}, \quad (11)$$

while in the number states representation we get

$$\begin{aligned} \langle n + d | \hat{K}_{\mu,\nu} | n \rangle &= \frac{1}{2\pi} z^2 e^{-izx} e^{-\frac{z^2}{4}(\mu^2 + \nu^2)} \\ &\times \sum_{l=0}^n \frac{\sqrt{n!(n+d)!}}{(n-l)!} \frac{[-\frac{z}{\sqrt{2}}(\nu - i\mu)]^l}{l!} \frac{[\frac{z}{\sqrt{2}}(\nu + i\mu)]^{(l+d)}}{(l+d)!}, \end{aligned} \quad (12)$$

and finally in the coordinate representation we obtain

$$\langle q' | \hat{K}_{\mu,\nu} | q'' \rangle = \frac{1}{2\pi} z^2 e^{-izx} e^{iz^2\mu\nu/2} e^{iz\mu q'} \delta(q'' - z\nu - q'). \quad (13)$$

If, and only if, the kernel operator is bounded every moment of the kernel is bounded for all possible distributions  $w(x, \mu, \nu)$ , then, according to the central limit theorem, the matrix elements of Eq. (9) can be sampled on a sufficiently large sets of data only in the number or coherent states basis, as can be evicted from the expressions (11), (12) and (13). That is in perfect agreement with the results of Ref. [10].

### 3 Examples

In this Section we will consider some examples of interest to check the validity of Eq. (9). We first consider the case of the harmonic oscillator's ground state for which we have [11]:

$$w(x, \mu, \nu) = [\pi(\mu + \nu)]^{-1/2} \exp\left[-\frac{x^2}{\mu^2 + \nu^2}\right]. \quad (14)$$

In the coherent states basis of Eq. (11) we obtain:

$$\int dx d\mu d\nu \quad w(x, \mu, \nu) \langle \alpha | \hat{K}_{\mu, \nu} | \beta \rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}}, \quad (15)$$

which exactly is the product  $\langle \alpha | 0 \rangle \langle 0 | \beta \rangle$  as we expected. As another example we consider the Schrödinger cat state [13] of the type discussed in Ref. [1]

$$|\Psi\rangle = \frac{|a + ib\rangle + |a - ib\rangle}{\{2[1 + \cos(2ab) \exp(-2b^2)]\}^{1/2}}, \quad (16)$$

with  $a$  and  $b$  arbitrary real numbers, for which we have a marginal distribution given by [11]

$$\begin{aligned} w(x, \mu, \nu) &= \left(\frac{2}{\pi}\right)^{1/2} \left[\frac{1}{\mu^2 + \nu^2}\right]^{1/2} \frac{1}{[1 + \cos(2ab) \exp(-2b^2)]} \exp\left[-2\frac{(x - \mu a)^2 + b^2 \nu^2}{\mu^2 + \nu^2}\right] \\ &\times \left\{ \cosh\left[\frac{4\nu b(x - \mu a)}{\mu^2 + \nu^2}\right] + \cos\left[\frac{2b(2\mu x - a(\mu^2 - \nu^2))}{\mu^2 + \nu^2}\right] \right\}. \end{aligned} \quad (17)$$

Then, from Eq. (9) we have in the coherent states basis of Eq. (11)

$$\begin{aligned} &\int dx d\mu d\nu \quad w(x, \mu, \nu) \langle \alpha | \hat{K}_{\mu, \nu} | \beta \rangle \\ &= \frac{e^{-(a^2 + b^2) - (|\alpha|^2 + |\beta|^2)/2}}{[1 + \cos(2ab)e^{-2b^2}]} [e^{(\alpha^* + \beta)(a + ib)} + e^{\alpha^*(a + ib) + \beta(a - ib)} + e^{\alpha^*(a - ib) + \beta(a + ib)} + e^{(\alpha^* + \beta)(a - ib)}] \end{aligned} \quad (18)$$

which exactly is the product  $\langle \alpha | \Psi \rangle \langle \Psi | \beta \rangle$  as we expect. Finally, we consider the application of Eq. (9) to the thermal equilibrium state of the oscillator at temperature  $T$ . In this case the Wigner function is given by [14]:

$$W_T(q, p) = 2 \tanh\left(\frac{\hbar\omega}{2kT}\right) \exp\left[-(q^2 + p^2) \tanh\left(\frac{\hbar\omega}{2kT}\right)\right]. \quad (19)$$

From this equation the marginal distribution can be easily obtained as follows [11]

$$\begin{aligned} w(x, \mu, \nu) &= \frac{1}{2\pi} \frac{1}{\mu} \int dp \quad W_T\left(\frac{x}{\mu} - \frac{\nu}{\mu} p, p\right) \\ &= \left[\frac{\Lambda}{\pi(\mu^2 + \nu^2)}\right]^{1/2} \exp\left[-\Lambda \frac{x^2}{\mu^2 + \nu^2}\right], \end{aligned} \quad (20)$$

where we have introduced  $\Lambda = \tanh\left(\frac{\hbar\omega}{2kT}\right)$ . Now, we use Eqs. (11) and (20) to calculate

$$\begin{aligned} &\int dx d\mu d\nu \quad w(x, \mu, \nu) \langle \alpha | \hat{K}_{\mu, \nu} | \beta \rangle \\ &= 2 \left[\frac{\Lambda}{1 + \Lambda}\right] \exp\left[\frac{1 - \Lambda}{1 + \Lambda} \alpha^* \beta - \frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}\right]. \end{aligned} \quad (21)$$

This result is the same that we could have by directly performing  $\langle \alpha | \hat{\rho}_T | \beta \rangle$  with the thermal density operator  $\hat{\rho}_T$  given by [14]

$$\hat{\rho}_T = (1 - e^{-\hbar\omega/kT}) \sum_n |n\rangle \langle n| e^{-n\hbar\omega/kT}. \quad (22)$$

## 4 Comparision with the usual tomographic technique

A relation between the density operator and the marginal distribution analogous to that of Eq. (9) can be derived starting from another operator identity such as [6]

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \text{Tr}\{\hat{\rho}\hat{D}(\alpha)\} D^{-1}(\alpha) \quad (23)$$

which, by the change of variables  $\mu = -\sqrt{2}\Im m \alpha$ ,  $\nu = \sqrt{2}\Re e \alpha$ , becomes

$$\hat{\rho} = \frac{1}{2\pi} \int d\mu d\nu \text{Tr}\{\hat{\rho}e^{-i\hat{X}}\} e^{i\hat{X}} = \frac{1}{2\pi} \int d\mu d\nu \text{Tr}\{\hat{\rho}e^{-i\hat{x}}\} e^{i\hat{x}} \quad (24)$$

where  $\hat{x} = \hat{X} - \delta$ . The trace can be now evaluated using the complete set of eigenvectors  $\{|x\rangle\}$  of the operator  $\hat{x}$ , obtaining

$$\text{Tr}\{\hat{\rho}e^{-i\hat{x}}\} = \int dx \ w(x, \mu, \nu) e^{-ix} \quad (25)$$

then, putting this one into Eq. (24), we have a relation of the same form of Eq. (9) with the kernel given by

$$\hat{K}_{\mu,\nu} = \frac{1}{2\pi} e^{-ix} e^{i\hat{x}} = \frac{1}{2\pi} e^{-ix} e^{i\mu\hat{q} + i\nu\hat{p}}, \quad (26)$$

which is the same of Eq. (10) setting  $z = 1$ . It means that we now have only one particular Fourier component due to the particular change of variables (the most general should be  $z\mu = -\sqrt{2}\Im m \alpha$  and  $z\nu = \sqrt{2}\Re e \alpha$ ).

In order to reconstruct the usual tomographic formula for the homodyne detection [10] we need to pass in polar variables, i.e.  $\mu = -r \cos \phi$ ,  $\nu = -r \sin \phi$ , then

$$\hat{x} \rightarrow -r\hat{x}_\phi = -r[\hat{q} \cos \phi + \hat{p} \sin \phi]. \quad (27)$$

Furthermore, denoting with  $x_\phi$  the eigenvalues of the quadrature operator  $\hat{x}_\phi$ , we have

$$\text{Tr}\{\hat{\rho}e^{-i\hat{x}}\} = \text{Tr}\{\hat{\rho}e^{ir\hat{x}_\phi}\} = \int dx_\phi \ w(x_\phi, \phi) e^{irx_\phi} \quad (28)$$

and thus, from Eq. (24)

$$\hat{\rho} = \int d\phi dx_\phi \quad w(x_\phi, \phi) \hat{K}_\phi \quad (29)$$

with

$$\hat{K}_\phi = \frac{1}{2\pi} \int dr \quad r e^{ir(x_\phi - \hat{x}_\phi)} \quad (30)$$

which is the same of Ref. [10]. Substantially, the kernel of Eq. (30) is given by the radial integral of the kernel of Eq. (26), and this is due to the fact that we go from a general transformation, with two free parameters, to a particular transformation (homodyne rotation) with only one free parameter and then we need to integrate over the other one.

## 5 Two-Mode Tomography

In this Section we discuss an extension of the above approach to multimode systems and, for simplicity and applications, we concentrate on the two-mode case. From now on we use the convention that the vector symbol represents only two dimensional vectors.

### 5.1 Two-Mode Quasiprobability and Marginal Distribution for Two Squeezed and Shifted Quadratures

We introduce the vector operators  $\hat{\vec{q}} = (\hat{q}_1, \hat{q}_2)$ ,  $\hat{\vec{p}} = (\hat{p}_1, \hat{p}_2)$  and then the following observables

$$(\hat{\vec{X}}, \hat{\vec{Y}}) = \Lambda (\hat{\vec{q}}, \hat{\vec{p}})^T + (\vec{\delta}, \vec{\delta}') \quad (31)$$

in which  $\Lambda$  is a real symplectic  $4 \times 4$  matrix

$$\Lambda \sigma \Lambda^T = \sigma; \quad \sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (32)$$

and  $(\vec{\delta}, \vec{\delta}') = (\delta_1, \delta_2, \delta'_1, \delta'_2)$  is a real c-number four-vector corresponding to the shifts of the four quadratures. Furthermore, the components  $X_1, X_2, Y_1, Y_2$  are related to the inhomogeneous symplectic group  $ISp(4, R)$ . Then we introduce the marginal distribution for the observable  $\hat{\vec{X}} = (\hat{X}_1, \hat{X}_2)$  due to relations which, obviously, are two-mode generalizations of the connection



between the characteristic function and the marginal distribution, given in Ref. [6] for one-mode case. We have firstly the characteristic function as

$$\chi(\vec{k}) = \langle e^{i\vec{k}\hat{X}} \rangle = \text{Tr}\{\hat{\rho}e^{i\vec{k}\hat{X}}\}; \quad \vec{k}\hat{X} = k_1\hat{X}_1 + k_2\hat{X}_2 \quad (33)$$

where  $\vec{k} = (k_1, k_2)$ , and  $\hat{\rho}$  is the density operator of the two-mode system. In terms of the Wigner function  $W(\vec{q}, \vec{p})$  of the system it is

$$\chi(\vec{k}) = \frac{1}{(2\pi)^2} \int e^{i\vec{k}\vec{X}} W(\vec{q}, \vec{p}) d\vec{q}d\vec{p}. \quad (34)$$

Here  $\hat{X}$  is, in principle, any observable, but to be concrete below we will consider this vector to have components

$$\begin{aligned} \hat{X}_1 &= \vec{\mu}\vec{q} + \vec{\nu}\vec{p} + \delta_1 \\ \hat{X}_2 &= \vec{\mu}'\vec{q} + \vec{\nu}'\vec{p} + \delta_2 \end{aligned} \quad (35)$$

with  $\vec{\mu} = (\Lambda_{11}, \Lambda_{12})$ ;  $\vec{\nu} = (\Lambda_{13}, \Lambda_{14})$ ;  $\vec{\mu}' = (\Lambda_{21}, \Lambda_{22})$ ;  $\vec{\nu}' = (\Lambda_{23}, \Lambda_{24})$ . The marginal distribution  $w(\vec{X}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}', \vec{\delta})$  of the vector variable  $\vec{X}$ , depending on the parameters of the symplectic transformation  $\Lambda$  and shift vector  $\vec{\delta}$ , is given by the relation

$$\begin{aligned} w(\vec{X}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}', \vec{\delta}) &= \frac{1}{(2\pi)^2} \int d\vec{k} \chi(\vec{k}) e^{-i\vec{k}\vec{X}} \\ &= \frac{1}{(2\pi)^4} \int d\vec{k} d\vec{q} d\vec{p} W(\vec{q}, \vec{p}) \exp[-ik_1(X_1 - \vec{\mu}\vec{q} - \vec{\nu}\vec{p} - \delta_1) \\ &\quad - ik_2(X_2 - \vec{\mu}'\vec{q} - \vec{\nu}'\vec{p} - \delta_2)]. \end{aligned} \quad (36)$$

From this formula it is clear that also in the two-mode case the marginal distribution depends on the difference  $\vec{X} - \vec{\delta} = \vec{x}$ , so that the replacement  $w(\vec{X}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}', \vec{\delta}) \rightarrow w(\vec{x}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}')$  is possible, with the normalization

$$\int w(\vec{x}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}') d\vec{x} = 1. \quad (37)$$

Our aim is to express the Wigner function  $W(\vec{q}, \vec{p})$  in terms of the marginal distribution probability  $w(\vec{x}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}')$ , i.e. to invert the formula (36). For this purpose we perform a Fourier transforms

of Eq. (36) and, after some minor algebra [11], we arrive at

$$W(-\vec{m}/z_1, -\vec{n}/z_1) = (2\pi)^4 z_1^4 e^{-i(\vec{\mu}'\vec{m} + \vec{\nu}'\vec{n})z_2/z_1} w(\vec{z}, \vec{m}, \vec{n}, \vec{\mu}', \vec{\nu}') \quad (38)$$

where the vectors  $\vec{m} = (m_1, m_2)$ ;  $\vec{n} = (n_1, n_2)$ ;  $\vec{z} = (z_1, z_2)$  are the conjugate variables to  $\vec{\mu}$ ,  $\vec{\nu}$  and  $\vec{x}$  respectively.

## 5.2 Two-Mode Quasiprobability and Marginal Distribution for One Squeezed and Shifted Quadrature

We will now discuss the connection between the Wigner function  $W(\vec{q}, \vec{p})$  and the marginal distribution  $\tilde{w}(x_1, \vec{\mu}, \vec{\nu})$  for only one quadrature  $\hat{X}_1$ . This latter is related to the previous by

$$\tilde{w}(x_1, \vec{\mu}, \vec{\nu}) = \int w(x_1, x_2, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}') dx_2. \quad (39)$$

Inserting in this one the expression of  $w$  given in Eq. (36), we may see that the marginal distribution  $\tilde{w}$  is connected with the characteristic function  $\tilde{\chi}(k_1) = \chi(k_1, k_2 = 0)$ ; where  $\chi(k_1, k_2)$  is given in Eq. (33). Repeating step by step the procedure that leads to Eq. (38) we have in this case

$$W(-\vec{m}/z_1, -\vec{n}/z_1) = (2\pi)^3 z_1^4 \tilde{w}(z_1, \vec{m}, \vec{n}). \quad (40)$$

This formula is similar to the one-mode case discussed in Ref. [11] and in the previous section, Eq. (7). The possibility to relate the Wigner function of the two-mode system with different marginal distributions,  $w(\vec{x})$  or  $\tilde{w}(x_1)$ , is connected with the use of different quantities of information obtainable from measurements. Of course, the two-dimensional distribution function contains much more experimental information than the one-dimensional. However, in both cases, information are overcomplete to determine the Wigner function of the quantum state, but must be selfconsistent. In concrete situations these varieties of descriptions might give a freedom to choose the convenient type of measurements.

## 6 Invariant Expression for Two-Mode Density Operator

We start from a generalization of Eq. (2) to the two-mode case

$$\hat{\rho} = \int \frac{d^2\alpha_1 d^2\alpha_2}{\pi^2} W(\vec{\alpha}) \hat{T}(\vec{\alpha}) \quad (41)$$

with  $\vec{\alpha} = (\alpha_1, \alpha_2)$  and  $\alpha_j = (q_j + ip_j)/\sqrt{2}$ ;  $j = 1, 2$ . Then inserting in this one the Wigner function of Eq. (38), and expressing the marginal distribution in terms of its Fourier transform, we have

$$\hat{\rho} = \int \frac{d\vec{q} d\vec{p}}{(2\pi)^4} \hat{T}(\vec{q}, \vec{p}) \int dxd\vec{\mu} d\vec{\nu} z_1^4 e^{iz_2(\vec{\mu}'\vec{q} + \vec{\nu}'\vec{p})} w(\vec{x}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}') e^{iz_1\vec{\mu}\vec{q} + iz_1\vec{\nu}\vec{p} - i\vec{z}\vec{x}} \quad (42)$$

that can be written as

$$\hat{\rho} = \int d\vec{x} d\vec{\mu} d\vec{\nu} w(\vec{x}, \vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}') \hat{K}_{\vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}'} \quad (43)$$

with the kernel operator explicitly given by

$$\begin{aligned} \hat{K}_{\vec{\mu}, \vec{\nu}, \vec{\mu}', \vec{\nu}'} &= \frac{1}{(2\pi)^2} z_1^4 e^{-i\vec{z}\vec{x}} \\ &\times \exp\left\{ \frac{1}{\sqrt{2}} [z_1(\vec{\nu} + i\vec{\mu}) + z_2(\vec{\nu}' + i\vec{\mu}')] \hat{\vec{a}} - \frac{1}{\sqrt{2}} [z_1(\vec{\nu} - i\vec{\mu}) + z_2(\vec{\nu}' - i\vec{\mu}')] \hat{\vec{a}}^\dagger \right\} \end{aligned} \quad (44)$$

with  $\hat{\vec{a}} = (\hat{a}_1, \hat{a}_2)$  the vector operator describing the two modes. In the case where the Wigner function is given by the marginal distribution of only one quadrature, as in Eq. (40), the same procedure leads to an expression similar to Eq. (43), but with a different kernel operator

$$\hat{K}_{\vec{\mu}, \vec{\nu}} = \frac{1}{(2\pi)^2} z_1^4 e^{-iz_1 x_1} \exp\left\{ \frac{z_1}{\sqrt{2}} [(\vec{\nu} + i\vec{\mu}) \hat{\vec{a}} - (\vec{\nu} - i\vec{\mu}) \hat{\vec{a}}^\dagger] \right\} \quad (45)$$

which is very similar to that of Eq. (10). It should be noticed that as a direct extension of the arguments of Sec. 2, also the kernels in Eqs. (44) and (45) are not bounded in the quadrature (or coordinate) representation, then in this case it is not possible to sample the density matrix elements.

## 7 Applications

Here we shall consider some applications of the presented tomographic scheme. We are aware that the crucial point might be the practical achievement of the generic linear quadrature such

as in Eq. (35). To consider, however, a measurement scheme as an optical implementation of the developed formalism, let us go back, at first, to the one dimensional case. The quadrature of Eq. (1) could be experimentally accessible by using for example the squeezing pre-amplification (pre-attenuation) of a field mode which is going to be measured (a similar method in different context was discussed in Ref. [15]). In fact, let  $\hat{a}$  be the signal field mode to be detected, when it passes through a squeezer it becomes  $\hat{a}_s = \hat{a} \cosh s - \hat{a}^\dagger e^{i\theta} \sinh s$ , where  $s$  and  $\theta$  characterize the complex squeezing parameter  $\zeta = se^{i\theta}$  [16]. Then, if we subsequently detect the field by using the balanced homodyne scheme, we get an output signal proportionnal to the average of the following quadrature

$$\hat{E}(\phi) = \frac{1}{\sqrt{2}}(\hat{a}_s e^{-i\phi} + \hat{a}_s^\dagger e^{i\phi}), \quad (46)$$

where  $\phi$  is the local oscillator phase. When this phase is locked to that of the squeezer, such that  $\phi = \theta/2$ , Eq. (46) becomes

$$\hat{E}(\phi) = \frac{1}{\sqrt{2}} \left( \hat{a} e^{-i\theta/2} [\cosh s - \sinh s] + \hat{a}^\dagger e^{i\theta/2} [\cosh s - \sinh s] \right), \quad (47)$$

which, essentially, coincides with Eq. (1), if one recognizes the independent parameters

$$\mu = [\cosh s - \sinh s] \cos(\theta/2); \quad \nu = [\cosh s - \sinh s] \sin(\theta/2). \quad (48)$$

The shift parameter  $\delta$  has not a real physical meaning, since it causes only a displacement of the distribution along the  $X$  line without changing its shape, as can be evicted from Eqs. (4) and (6). So, in a practical situation it can be omitted.

In the two-mode case it could be interesting to use the connection between the Wigner function and the marginal distribution of only one quadrature in the case of heterodyne detection (particular used to detect multimode squeezed states). We may refer to this scheme as the optical heterodyne tomography. In fact, in balanced heterodyne detection the measured current is determined by [16]

$$\hat{E}(\phi) = \frac{E_1}{\sqrt{2}} [e^{i\phi} \hat{a}_1^\dagger + e^{-i\phi} \hat{a}_1] + \frac{E_2}{\sqrt{2}} [e^{i\phi} \hat{a}_2^\dagger + e^{-i\phi} \hat{a}_2]. \quad (49)$$

If one uses squeezed pre-amplification (pre-attenuation) for each single mode, as discussed above, both  $E_1$  and  $E_2$  are independent variables, however, we now have only one phase, the local oscillator phase  $\phi$ ; thus, in order to have a fourth independent variable which would allow a complete reconstruction of two-mode quasi-probability, we could consider the phase shifter used in the first step of the measurement procedure. Since the two modes have different frequencies, the phase shifter induces different phase changes in the two modes; for example

$$\hat{a}_1 \rightarrow \hat{a}_1 e^{i\theta_1}; \quad \hat{a}_2 \rightarrow \hat{a}_2 e^{i\theta_2}; \quad (50)$$

where the difference  $\theta_1 - \theta_2$  depends on the length of the optical path in the phase shifter. Upon using this requirement, the measured quadrature (49) will become

$$\hat{E}(\phi) = \frac{E_1}{\sqrt{2}}[e^{i(\phi+\theta_1)}\hat{a}_1^\dagger + e^{-i(\phi+\theta_1)}\hat{a}_1] + \frac{E_2}{\sqrt{2}}[e^{i(\phi+\theta_2)}\hat{a}_2^\dagger + e^{-i(\phi+\theta_2)}\hat{a}_2], \quad (51)$$

that corresponds to the quadrature  $X_1$  of Eq. (35) with  $\vec{\mu} = (E_1 \cos(\phi + \theta_1), E_2 \cos(\phi + \theta_2))$  and  $\vec{\nu} = (E_1 \sin(\phi + \theta_1), E_2 \sin(\phi + \theta_2))$ . For the shift parameter  $\delta_1$  the previous considerations, made in one mode case, still hold.

We would now present some examples of the probability  $\tilde{w}$  measurable with the above scheme. We first consider the most general two-mode squeezed states described by the Wigner function of the form [17]

$$W(\vec{q}, \vec{p}) = (\det \mathcal{M})^{-1/2} \exp\left[-\frac{1}{2}\left((\vec{q}, \vec{p}) - \langle(\hat{\vec{q}}, \hat{\vec{p}})\rangle\right)\mathcal{M}^{-1}\left((\vec{q}, \vec{p}) - \langle(\hat{\vec{q}}, \hat{\vec{p}})\rangle\right)^T\right] \quad (52)$$

where the real symmetric dispersion matrix  $\mathcal{M}$  has ten variances

$$\mathcal{M}_{\alpha\beta} = \frac{1}{2}\langle(\hat{\vec{q}}, \hat{\vec{p}})_\alpha(\hat{\vec{q}}, \hat{\vec{p}})_\beta + (\hat{\vec{q}}, \hat{\vec{p}})_\beta(\hat{\vec{q}}, \hat{\vec{p}})_\alpha\rangle - \langle(\hat{\vec{q}}, \hat{\vec{p}})_\alpha\rangle\langle(\hat{\vec{q}}, \hat{\vec{p}})_\beta\rangle \quad \alpha, \beta = 1, 2, 3, 4. \quad (53)$$

By integrating only on  $k_1$  with  $k_2 = 0$ , we obtain the marginal distribution of only one quadrature  $X_1$  as given in Eq. (36)

$$\begin{aligned} \tilde{w}(x_1, \vec{\mu}, \vec{\nu}) &= \frac{1}{(2\pi)^3} \int dk_1 \int d\vec{q} d\vec{p} W(\vec{q}, \vec{p}) e^{-ik_1(x_1 - \vec{\mu}\vec{q} - \vec{\nu}\vec{p})} \\ &= [2\pi(\vec{\mu}, \vec{\nu})\mathcal{M}(\vec{\mu}, \vec{\nu})^T]^{-1/2} \exp\left[-\frac{x_1^2}{2(\vec{\mu}, \vec{\nu})\mathcal{M}(\vec{\mu}, \vec{\nu})^T}\right] \end{aligned} \quad (54)$$

where we have assumed  $\langle (\hat{\vec{q}}, \hat{\vec{p}})_\alpha \rangle = 0$  for all  $\alpha$ . This distribution is a Gaussian for the quadrature variable.

As a second example, we consider the two-mode Schrödinger-cat state [18] defined as

$$|\vec{A}_\pm\rangle = N_\pm(|\vec{A}\rangle \pm |-\vec{A}\rangle) \quad (55)$$

with

$$N_+ = \frac{e^{|\vec{A}|^2/2}}{2\sqrt{\cosh |\vec{A}|^2}}; \quad N_- = \frac{e^{|\vec{A}|^2/2}}{2\sqrt{\sinh |\vec{A}|^2}}. \quad (56)$$

The Wigner function of this state is

$$\begin{aligned} W_{\vec{A}_\pm}(\vec{q}, \vec{p}) = N_\pm^2 \quad [ & W_{(\vec{A}, \vec{B}=\vec{A})}(\vec{q}, \vec{p}) \pm W_{(\vec{A}, \vec{B}=-\vec{A})}(\vec{q}, \vec{p}) \\ & \pm W_{(-\vec{A}, \vec{B}=\vec{A})}(\vec{q}, \vec{p}) \pm W_{(-\vec{A}, \vec{B}=-\vec{A})}(\vec{q}, \vec{p})] \end{aligned} \quad (57)$$

where

$$W_{\vec{A}, \vec{B}} = 4 \exp \left[ -2\vec{\alpha}\vec{\alpha}^* + 2\vec{A}\vec{\alpha}^* + 2\vec{B}^*\vec{\alpha} - \vec{A}\vec{B}^* - \frac{|\vec{A}|^2}{2} - \frac{|\vec{B}|^2}{2} \right] \quad (58)$$

is the Wigner function for two-mode coherent state [19] with

$$\vec{\alpha} = \frac{\vec{q} + i\vec{p}}{\sqrt{2}}. \quad (59)$$

Let us now consider the even coherent state, then to get the marginal distribution, we use, as in the previous case, Eq. (36) obtaining

$$\begin{aligned} \tilde{w}(x_1, \vec{\mu}, \vec{\nu}) = & \frac{2\pi^{-1/2}N_+^2}{\sqrt{|\vec{\mu}|^2 + |\vec{\nu}|^2}} \exp \left[ \frac{-x_1^2 - (\nu_1 P_1 + \mu_1 Q_1)^2 - (\nu_2 P_2 + \mu_2 Q_2)^2}{|\vec{\mu}|^2 + |\vec{\nu}|^2} \right] \\ \times & \left\{ \exp \left[ \frac{-(P_1^2 + Q_1^2)(\nu_2^2 + \mu_2^2) - (P_2^2 + Q_2^2)(\nu_1^2 + \mu_1^2) + 2(\mu_1 P_1 - \nu_1 Q_1)(\mu_2 P_2 - \nu_2 Q_2)}{|\vec{\mu}|^2 + |\vec{\nu}|^2} \right] \right. \\ \times & \cos \left[ \frac{2(\mu_1 P_1 + \mu_2 P_2 - \nu_1 Q_1 - \nu_2 Q_2)x_1}{|\vec{\mu}|^2 + |\vec{\nu}|^2} \right] \\ + & \left. \exp \left[ \frac{-2(\nu_1 P_1 + \mu_1 Q_1)(\nu_2 P_2 + \mu_2 Q_2)}{|\vec{\mu}|^2 + |\vec{\nu}|^2} \right] \cosh \left[ \frac{2(\nu_1 P_1 + \nu_2 P_2 + \mu_1 Q_1 + \mu_2 Q_2)x_1}{|\vec{\mu}|^2 + |\vec{\nu}|^2} \right] \right\}. \end{aligned} \quad (60)$$

where we have set

$$\vec{A} = \frac{\vec{Q} + i\vec{P}}{\sqrt{2}}. \quad (61)$$

In the case of  $\mu_2 = \nu_2 = 0$  and  $Q_2 = P_2 = 0$ , the marginal distribution of Eq. (60) reduces to the one-dimensional marginal distribution of Eq. (32) of Ref. [11]<sup>2</sup>.

## 8 Conclusions

We have shown that the optical tomography may be extended to two-mode case, and the new observables introduced, which are generic linear forms in quadratures, give the possibility of using various measurements, different from the usual ones, to determine the Wigner function of the system under study in terms of the marginal distributions. To better understand the meaning of the measure of the observable  $\hat{X}$  of Eq. (1), we recall that this symplectic transformation could be represented as a composition of shift, rotation and squeezing. Along this line we have presented some schemes which could be implemented in optics. To be more precise, the shift parameter does not play a real physical role in the measurement process, it has been introduced for formal completeness and it expresses the possibility to achieve the desired marginal distribution by performing the measurements in an ensemble of frames which are each other shifted; (related method was early discussed in Ref. [20]). In an electro-optical system this only means to have the freedom of using different photocurrent scales in which the zero is shifted by a known amount.

The tomographic formalism is presented here in an invariant form for the density operator and it may be suitable for further numerical analysis. It is worth to remark that all the formulae of two-mode case may be obviously rewritten for N-mode case with arbitrary N. The presented extension gives the possibility to include the procedure of heterodyne detection (two-mode case) as well. Furthermore, the extension of the tomographic technique to multimode systems could also be useful to obtain the (complete) quantum information about a system in an indirect way [21].

Another conclusive remark is related to the group structure of the presented tomographic extension. Using generic observables which are linear in quadratures, we apply a symplectic group transform to an initial quadrature component. In one mode optical tomography this transform

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<sup>2</sup>Up to a misprint in this reference.

belongs to the rotation subgroup of the symplectic group  $ISp(2, R)$ . Thus the presented construction of the Wigner function (density operator) may be reformulated as the group problem for a particular symplectic group and we could call our extension as "symplectic optical tomography". On the other hand the scheme might be generalized to other Lie groups different from the symplectic one. We will develop these points in future papers.

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